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## Charged free fermions, vertex operators and the classical theory of conjugate nets

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**Abstract.** We show that the quantum field theoretical formulation of the  $\tau$ -function theory has a geometrical interpretation within the classical transformation theory of conjugate nets. In particular, we prove that (i) the partial charge transformations preserving the neutral sector are Laplace transformations, (ii) the basic vertex operators are Lévy and adjoint Lévy transformations and (iii) the diagonal soliton vertex operators generate fundamental transformations. We also show that the bilinear identity for the multicomponent Kadomtsev–Petviashvili hierarchy becomes, through a generalized Miwa map, a bilinear identity for the multidimensional quadrilateral lattice equations.

### 1. Introduction

The notion of  $\tau$ -function is a cornerstone in the theory of integrable systems since its discovery in 1976 by Hirota [17]. It allows us to reformulate many integrable equations as bilinear equations for  $\tau$ -functions and provides suitable methods for finding soliton solutions. In a series of papers [33], Sato *et al* introduced the concept of  $\tau$ -function in the framework of quantum field theory of free fermions in two-dimensional space-time. They were motivated by the fact that certain limits of correlation functions of the two-dimensional Ising model provide solutions of the Painlevé III equation [20]. As a result, they revealed an unexpected link with the isomonodromic deformation theory of linear differential equations, a classical mathematical subject started by Riemann in the last century and investigated by Schlesinger [36], Fuchs [15] and Garnier [16] among others. Later on [22], it was shown that the same quantum field theoretical formulation applies equally well for describing the Kadomtsev–Petviashvili (KP) hierarchy and its multicomponent extension in terms of  $b$ - $c$  systems of ghosts fields. This description turned out to be related with certain aspects of string theory as, for example, the connection between bosonic string amplitudes and Hirota's difference equation [32, 30, 18]. Furthermore, the Grassmannian model [35] of the  $\tau$ -function theory is strongly related with the operator formalism in string theory and conformal field theory [38, 2]. In this context, it is worth mentioning that the Korteweg–de Vries hierarchy (a reduction of KP) has appeared in a non-perturbative description of two-dimensional quantum gravity [6].

On the other hand, the theory of conjugate nets [7, 3, 13] is a classical subject in differential geometry developed by distinguished geometers of the last century and the beginning of the present one (Gauss, Lamé, Bonnet, Cayley, Bianchi, Darboux and Eisenhart). In particular, the transformation theory of conjugate nets is a well-established subject [13]. The orthogonal reduction is strongly connected with the theory of quasilinear systems of hydrodynamic type, which in the two-dimensional case includes the Euler equations of an incompressible fluid. Riemann devoted part of his work to the classification of these systems in terms of Riemann invariants, this was extended to the multidimensional case by Tsarev [37] showing that the classification problem of Hamiltonian systems of hydrodynamic type is equivalent to the study of orthogonal nets. Let us also mention that conjugate nets are connected with the description of the three wave resonant interaction. The discrete analogues of the conjugate nets and quadrilateral lattices are central objects of the integrable discrete geometry which is developing nowadays [10, 11]. Finally, Egoroff systems (a particular type of conjugate nets) have recently found application in topological field theory; namely, in the resolution of indescomposable Witten–Dijkgraaf–Verlinde–Verlinde associativity equations [12].

The aim of this paper is to show that the basic operations associated with the multicomponent KP theory have a distinguished geometric interpretation in terms of conjugate nets, their transformations and discretizations. Thus, the quantum field theoretical scheme introduced by the Kyoto school is strongly tied up not only with monodromy problems and integrable systems but also with the developments that started last century in the arena of classical differential geometry.

The layout of the paper is as follows, the first two sections have a introductory character: in section 2, we introduce standard material on the quantum field theory description of the multicomponent KP hierarchy (for an alternative approach see [21, 5]) and, in section 3, the theory of conjugate nets and its transformations, namely Laplace, Lévy, adjoint Lévy and fundamental transformations. The next two sections, section 4 and section 5, contains the main results of our paper. In section 4 we show the following.

- (1) The partial charge transformations preserving total charge, i.e. the Schlesinger transformations, are Laplace transformations.
- (2) The basic vertex operators can be identified with the Lévy and adjoint Lévy transformations.
- (3) The diagonal soliton vertex operator generates a fundamental transformation.

In this manner a complete list of equivalences among the basic transformations of both schemes arises. We underline that these correspondences are linked with a series of Fay identities for the  $\tau$ -function.

Finally, in section 5, we extend the Miwa transformation to the multicomponent case to obtain the quadrilateral lattice. This result, as we shall exhibit, has a natural geometrical interpretation.

## 2. $b$ - $c$ systems and $\tau$ -functions

The  $b$ - $c$  system of quantum fields, which appears as the system of ghost fields in string theory, is constructed in terms of the anticommutation relations

$$\begin{aligned} \{b_i(z), c_j(z')\} &= \delta_{ij} \delta(z - z') \\ \{b_i(z), b_j(z')\} &= \{c_i(z), c_j(z')\} = 0 \end{aligned}$$

where  $b_i(z)$  and  $c_i(z)$ ,  $i = 1, \dots, N$ , are free charged fermion fields defined on the unit circle  $S^1$ , and  $\delta(z - z')$  is the Dirac distribution on  $S^1$ .

The Clifford algebra generated by the  $b$ - $c$  system admits a representation in terms of bosonic variables. In this representation the fields act on the Fock space  $\mathcal{H}$  of complex-valued functions

$$\tau = \sum_{\ell} \tau(\ell, \mathbf{t}) \boldsymbol{\lambda}^{\ell}$$

with

$$\begin{aligned} \ell &:= (\ell_1, \dots, \ell_N) \in \mathbb{Z}^N \\ \mathbf{t} &:= (\mathbf{t}_1, \dots, \mathbf{t}_N) \in \mathbb{C}^{N \cdot \infty} & \mathbf{t}_i &:= (t_{i,1}, t_{i,2}, \dots) \in \mathbb{C}^{\infty} \\ \boldsymbol{\lambda} &:= (\lambda_1, \dots, \lambda_N) \in \mathbb{C}^N & \boldsymbol{\lambda}^{\ell} &:= \lambda_1^{\ell_1} \dots b \lambda_N^{\ell_N}. \end{aligned}$$

The representation of the  $b$ - $c$  generators takes the form [9]:

$$b_i(z) := \Gamma_i(z) S_i(z) \prod_{j>i} \pi_j \quad c_i(z) := \Gamma_i^*(z) S_i^*(z) \prod_{j>i} \pi_j \quad i = 1, \dots, N$$

where

$$\begin{aligned} \Gamma_i(z) &:= \exp(\xi(z, \mathbf{t}_i)) \mathbb{V}_i^-(z) & \Gamma_i^*(z) &:= \exp(-\xi(z, \mathbf{t}_i)) \mathbb{V}_i^+(z) \\ S_i(z) &:= \lambda_i z^{\lambda_i \partial / \partial \lambda_i} & S_i^*(z) &:= \frac{1}{\lambda_i} z^{1 - \lambda_i \partial / \partial \lambda_i} \\ \pi_i(\boldsymbol{\lambda}^{\ell}) &= (-1)^{\ell_i} \boldsymbol{\lambda}^{\ell}. \end{aligned}$$

Here we are denoting

$$\xi(z, \mathbf{t}_i) := \sum_{n=1}^{\infty} z^n t_{i,n}$$

and  $\mathbb{V}_i^{\pm}$  are operators defined by

$$\mathbb{V}_i^{\pm}(z) f(\mathbf{t}) := f(\mathbf{t} \pm [1/z] \mathbf{e}_i) \quad [1/z] := \left( \frac{1}{z}, \frac{1}{2z^2}, \frac{1}{3z^3}, \dots \right)$$

being  $\{\mathbf{e}_i\}_{i=1}^N$  the canonical generators of  $\mathbb{C}^N$ .

Alternatively, the action of the  $b$ - $c$  system can be formulated as

$$\begin{aligned} b_i(z) \tau(\ell, \mathbf{t}) &:= (-1)^{\sum_{j>i} \ell_j} z^{\ell_i - 1} \exp(\xi(z, \mathbf{t}_i)) \tau(\ell - \mathbf{e}_i, \mathbf{t} - [1/z] \mathbf{e}_i) \\ c_i(z) \tau(\ell, \mathbf{t}) &:= (-1)^{\sum_{j>i} \ell_j} z^{-\ell_i} \exp(-\xi(z, \mathbf{t}_i)) \tau(\ell + \mathbf{e}_i, \mathbf{t} + [1/z] \mathbf{e}_i). \end{aligned}$$

The Fock space decomposes into a direct sum of charge sectors

$$\mathcal{H} = \bigoplus_{q \in \mathbb{Z}} \mathcal{H}_q \quad \mathcal{H}_q = \{ \tau \in \mathcal{H} : Q \tau = q \tau \}$$

where the total charge operator  $Q := \sum_{i=1}^N Q_i$  is the sum of  $N$  commuting partial charges  $Q_i = \lambda_i \partial / \partial \lambda_i$ ,  $i = 1, \dots, N$ ; they correspond to the  $N$  different flavours of fermions of the model.

The  $N$ -component KP hierarchy can be formulated as the following bilinear identity

$$\mathcal{B}(\tau \otimes \tau) = 0 \quad \tau \in \mathcal{H}_0 \tag{1}$$

where

$$\mathcal{B} := \int_{S^1} \frac{dz}{z} \sum_{i=1}^N b_i(z) \otimes c_i(z).$$

In terms of the components  $\tau(\ell, \mathbf{t})$  we have

$$\int_{S^1} dz \sum_{m=1}^N (-1)^{\sum_{j>m} \ell_j + \ell'_j} \exp(\xi(z, \mathbf{t}_m - \mathbf{t}'_m)) z^{\ell_m - \ell'_m - 2} \times \tau(\ell - \mathbf{e}_m, \mathbf{t} - [1/z]\mathbf{e}_m) \tau(\ell' + \mathbf{e}_m, \mathbf{t}' + [1/z]\mathbf{e}_m) = 0 \tag{2}$$

for any  $\mathbf{t}, \mathbf{t}'$  and  $\ell, \ell'$  such that  $\ell_1 + \dots + \ell_N - 1 = \ell'_1 + \dots + \ell'_N + 1 = 0$ .

Let us mention that, as is well known, in the fermionic picture of the  $\tau$ -function one can define a vacuum in such a manner that the  $\tau$ -function becomes the vacuum expectation value of a suitable element of the Clifford algebra [9].

It can be shown that

$$[\Gamma_i \otimes \Gamma_i, \mathcal{B}] = [\Gamma_i^* \otimes \Gamma_i^*, \mathcal{B}] = 0$$

i.e., for any solution  $\tau$  of (1) the functions  $\Gamma_i \tau$  and  $\Gamma_i^* \tau$  satisfy (1) as well. Hence, the vertex operators  $\Gamma_i$  and  $\Gamma_i^*, i = 1, \dots, N$ , constitute a set of symmetries of (1).

In our subsequent analysis we will fix a given  $\ell$  and denote

$$\begin{aligned} \tau(\mathbf{t}) &:= \tau(\ell, \mathbf{t}) \\ \tau_{ij}(\mathbf{t}) &:= \mathcal{S}_{ij} \tau(\mathbf{t}) := \tau(\ell + \mathbf{e}_i - \mathbf{e}_j, \mathbf{t}). \end{aligned}$$

Observe that the vectors  $\alpha_{ij} = \mathbf{e}_i - \mathbf{e}_j$  are the roots of the  $A_{N-1}$  root system, so that any linear combination of them with integer coefficients is a point in the corresponding root lattice. The shift operators  $\mathcal{S}_{ij}$  along the root lattice vectors  $\alpha_{ij}$  correspond to the so-called Schlesinger transformations in monodromy theory [8, 19, 26] and satisfy the following relations

$$\mathcal{S}_{ij} \circ \mathcal{S}_{ji} = \text{id} \tag{3}$$

$$\mathcal{S}_{ij} \circ \mathcal{S}_{jk} = \mathcal{S}_{ik} \quad \mathcal{S}_{ij} \circ \mathcal{S}_{ki} = \mathcal{S}_{kj}. \tag{4}$$

This root lattice models all the possible transformations in the partial charges that preserves the total charge. Moreover,  $\mathcal{S}_{ij}$  are symmetries of (1).

The  $N \times N$  matrix Baker function  $\psi$  and its adjoint  $\psi^*$  can be defined in terms of the  $\tau$  function as

$$\begin{aligned} \psi_{ij}(z, \mathbf{t}) &= \varepsilon_{ij} z^{\delta_{ij}-1} \frac{\tau_{ij}(\mathbf{t} - [1/z]\mathbf{e}_j)}{\tau(\mathbf{t})} \exp(\xi(z, \mathbf{t}_j)) \\ \psi_{ij}^*(z, \mathbf{t}) &= \varepsilon_{ji} z^{\delta_{ij}-1} \exp(-\xi(z, \mathbf{t}_i)) \frac{\tau_{ij}(\mathbf{t} + [1/z]\mathbf{e}_i)}{\tau(\mathbf{t})} \end{aligned} \tag{5}$$

where  $\varepsilon_{ij} := \text{sgn}(j - i), j \neq i$  ( $\varepsilon_{ii} := 1$ ).

Observe that we have

$$\begin{aligned} \psi(z, \mathbf{t}) &:= \chi(z, \mathbf{t}) \psi_0(z, \mathbf{t}) \\ \psi^*(z, \mathbf{t}) &:= \psi_0(z, \mathbf{t})^{-1} \chi^*(z, \mathbf{t}) \end{aligned} \tag{6}$$

where  $\psi_0(z, \mathbf{t}) = \text{diag}(\exp(\xi(z, \mathbf{t}_1)), \dots, \exp(\xi(z, \mathbf{t}_N)))$ , and  $\chi$  and  $\chi^*$  are the bare Baker functions with the following asymptotic expansion

$$\begin{aligned} \chi(z) &\sim 1 + \beta z^{-1} + \mathcal{O}(z^{-2}) & z \rightarrow \infty \\ \chi^*(z) &\sim 1 - \beta z^{-1} + \mathcal{O}(z^{-2}) & z \rightarrow \infty \end{aligned} \tag{7}$$

and the matrix  $\beta$  is given by

$$\begin{aligned} \beta_{ii}(\mathbf{t}) &:= -\frac{\partial \ln \tau(\mathbf{t})}{\partial u_i} & i = 1, \dots, N \\ \beta_{ij}(\mathbf{t}) &:= \varepsilon_{ij} \frac{\tau_{ij}(\mathbf{t})}{\tau(\mathbf{t})} & i \neq j \quad i, j = 1, \dots, N \end{aligned} \tag{8}$$

with  $u_k := t_{k,1}, k = 1, \dots, N$ .

Thus, by setting  $\ell \rightarrow \ell + e_i$  and  $\ell' \rightarrow \ell - e_j$  in (2) one obtains

$$\int_{S^1} dz \psi(z, \mathbf{t}) \psi^*(z, \mathbf{t}') = 0. \tag{9}$$

The symmetry operators  $\Gamma_i$  and  $\Gamma_i^*$  of (1) induce a corresponding action, say  $\mathbb{G}_i$  and  $\mathbb{G}_i^*$ , respectively, on Baker functions:

$$\begin{aligned} \mathbb{G}_i(p) \psi(z, \mathbf{t}) [\mathbb{V}_i^-(p) \psi(z, \mathbf{t})] \left(-\frac{p}{z}\right)^{P_i} \\ \mathbb{G}_i^*(p) \psi(z, \mathbf{t}) = [\mathbb{V}_i^+(p) \psi(z, \mathbf{t})] \left(-\frac{z}{p}\right)^{P_i}. \end{aligned}$$

Here  $P_i$  stands for the matrix with elements  $(P_i)_{jk} = \delta_{ij} \delta_{ik}$ . Notice that in order that  $\mathbb{V}_i^\pm \psi(z, \mathbf{t})$  be convergent it is required that  $|p| > |z|$ .

Both bilinear identities (2) and (9) are useful for characterizing the  $N$ -component KP hierarchy. In particular, (9) is suitable for formulating the Grassmannian approach to the hierarchy [35], which in turn is very convenient in the derivation of the linear system of equations for the Baker functions. Let us, for instance, outline this approach for the Baker function  $\psi$ . To this end, we denote by  $W$  the set of  $N \times N$  matrix functions  $\varphi(z)$  such that:

$$\int_{S^1} dz \varphi(z) \psi^*(z, \mathbf{t}') = 0$$

for all  $\mathbf{t}'$  in the definition domain of  $\psi^*$ . Under appropriate conditions the set  $W$  belongs to an infinite-dimensional Grassmannian manifold [35]. From (9) it follows that  $W$  is a left  $M_N(\mathbb{C})$ -module, with  $M_N(\mathbb{C})$  being the ring of  $N \times N$  complex matrices. We shall use the standard notation  $E_{ij}$  for the linear basis in  $M_N(\mathbb{C})$ , and in particular  $P_i = E_{ii}$ . As a consequence of (9) and the form of the asymptotic expansion of  $\psi^*$  as  $z \rightarrow \infty$  one has that for any  $\mathbf{t}$ :

$$W = \bigoplus_{n \geq 0} M_N(\mathbb{C}) \cdot v_n(\mathbf{t}) \quad v_n(z, \mathbf{t}) = \left( \sum_{k=1}^N \frac{\partial}{\partial u_k} \right)^n \psi(z, \mathbf{t}). \tag{10}$$

Notice that

$$v_n(z) \sim (z^n + \mathcal{O}(z^{n-1})) \psi_0(z) \quad z \rightarrow \infty. \tag{11}$$

Thus, the linear system for the  $N$ -component KP hierarchy results from the decompositions of the time derivatives of  $\psi$  in terms  $v_n, n = 0, \dots, \infty$ . In particular, by decomposing  $P_i \partial \psi / \partial u_k, i \neq k$ , one gets

$$\frac{\partial \psi_i}{\partial u_k} = \beta_{ik} \psi_k \tag{12}$$

with

$$\psi_i := (\psi_{i1}, \dots, \psi_{iN}).$$

Proceeding in a similar way for the adjoint Baker function we arrive at

$$\frac{\partial \psi_j^*}{\partial u_k} = \psi_k^* \beta_{kj} \quad j \neq k \tag{13}$$

where

$$\psi_i^* := \begin{pmatrix} \psi_{1i}^* \\ \vdots \\ \psi_{Ni}^* \end{pmatrix}.$$

The compatibility of either (12) or (13) leads to the Darboux equations for the  $\beta$ 's:

$$\frac{\partial \beta_{ij}}{\partial u_k} = \beta_{ik} \beta_{kj} \quad i, j \text{ and } k \text{ different.} \quad (14)$$

As for the bilinear identity (2), the evaluation of the residue at infinity of the integrand provides the Hirota representation of the  $N$ -component KP hierarchy. In particular,

$$\tau \frac{\partial^2 \tau}{\partial u_i \partial u_j} - \frac{\partial \tau}{\partial u_i} \frac{\partial \tau}{\partial u_j} - \tau_{ij} \tau_{ji} = 0 \quad i \neq j \quad (15)$$

$$\tau \frac{\partial \tau_{ij}}{\partial u_k} - \tau_{ij} \frac{\partial \tau}{\partial u_k} - \varepsilon_{ij} \varepsilon_{ik} \varepsilon_{kj} \tau_{ik} \tau_{kj} = 0 \quad i, j \text{ and } k \text{ different} \quad (16)$$

being (16) the Hirota form of the above Darboux equations.

By setting  $\ell \rightarrow \ell + e_i$  and  $\ell' \rightarrow \ell + e_k - e_l - e_j$  in (2) we obtain

$$\varepsilon_{ij} \varepsilon_{kl} \tau \mathcal{S}_{ik}(\tau_{jl}) + \varepsilon_{il} \varepsilon_{jk} \tau_{ik} \tau_{jl} - \varepsilon_{ik} \varepsilon_{jl} \tau_{il} \tau_{jk} = 0 \quad i, j, k \text{ and } l \text{ different.} \quad (17)$$

This relation, which can be found in [21], is just a Fay trisecant formula for theta functions on Riemann surfaces [28].

### 3. Conjugate nets and quadrilateral lattices

The Darboux equations (14) for the so-called rotation coefficients  $\beta_{ij}$  characterize  $N$ -dimensional submanifolds of  $\mathbb{R}^M$ ,  $N \leq M$ , parametrized by conjugate coordinate systems (multiconjugate nets) [7], and are the compatibility conditions of the following linear system:

$$\frac{\partial \mathbf{X}_j}{\partial u_i} = \beta_{ji} \mathbf{X}_i \quad i, j = 1, \dots, N \quad i \neq j \quad (18)$$

involving  $M$ -dimensional vectors  $\mathbf{X}_i$ , tangent to the coordinate lines. The so-called Lamé coefficients  $H_i$  satisfy

$$\frac{\partial H_j}{\partial u_i} = \beta_{ij} H_i \quad i, j = 1, \dots, N \quad i \neq j$$

in terms of which the points  $x$  of the net are found by integrating the following equation

$$\frac{\partial x}{\partial u_i} = \mathbf{X}_i H_i \quad i = 1, \dots, N.$$

Thus, given the Baker function  $\psi$ , one can construct conjugate nets with  $\beta$ 's as appearing in (7) and with the tangent vectors  $\mathbf{X}_i$  being the rows of the matrix

$$X(t) = \int_{S^1} dz \psi(z, t) f(z) \quad (19)$$

for some distribution matrix  $f(z) \in M_{N \times M}(\mathbb{C})$ . Given the adjoint Baker function  $\psi^*$ , the Lamé coefficients are given by the entries of the row matrix

$$H(t) = \int_{S^1} dz g(z) \psi^*(z, t) \quad i = 1, \dots, N \quad (20)$$

for some distribution row matrix  $g(z) \in \mathbb{C}^N$ .

Therefore, we arrive at the following proposition.

**Proposition 1.** *The solutions of the  $N$ -component KP hierarchy describe  $N$  dimensional conjugate nets with coordinates  $u_i = t_{i,1}$ ,  $i = 1, \dots, N$ , while the remaining times  $t_{i,k}$ , for  $k > 1$ , describe integrable iso-conjugate deformations of the nets.*

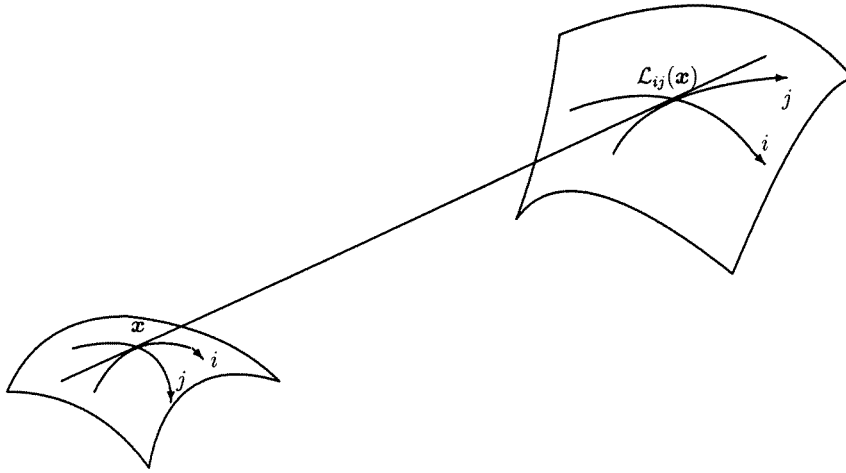


Figure 1. The Laplace transformation.

In particular, for  $N = 2$ , the Davey–Stewartson hierarchy describes the iso-conjugate deformations of two-dimensional conjugate nets [23].

Transformations of conjugate nets have been extensively studied in the literature [13] and the most convenient way to characterize them is through the notion of congruences of lines. The basic transformations of conjugate nets are listed below.

(i) The Laplace transformation  $\mathcal{L}_{ij}(x)$ ,  $i \neq j$ , of a conjugate net  $x$  is the  $j$ th focal net of the  $i$ th tangent congruence of  $x$  [7]; in simple terms it means that the line tangent to the  $i$ th coordinate line at a point  $x$  of the net is tangent to the  $j$ th coordinate line of the transformed net at the corresponding point  $\mathcal{L}_{ij}(x)$  (see figure 1).

It turns out [7, 11] that the position points of the transformed net are given by

$$\mathcal{L}_{ij}(x) = x - \frac{H_j}{\beta_{ij}} \mathbf{X}_i.$$

The corresponding transformation for the rotation coefficients  $\beta_{ij}$  are [14]

$$\mathcal{L}_{ij}(\beta_{ij}) = \beta_{ij} \left( \beta_{ij} \beta_{ji} - \frac{\partial^2 \log \beta_{ij}}{\partial u_i \partial u_j} \right) \tag{21}$$

$$\mathcal{L}_{ij}(\beta_{ji}) = \frac{1}{\beta_{ij}} \tag{22}$$

$$\mathcal{L}_{ij}(\beta_{ki}) = \frac{\beta_{kj}}{\beta_{ij}} \tag{23}$$

$$\mathcal{L}_{ij}(\beta_{jk}) = -\frac{\beta_{ik}}{\beta_{ij}} \tag{24}$$

$$\mathcal{L}_{ij}(\beta_{ik}) = -\beta_{ij} \frac{\partial}{\partial u_i} \left( \frac{\beta_{ik}}{\beta_{ij}} \right) \tag{25}$$

$$\mathcal{L}_{ij}(\beta_{kj}) = \beta_{ij} \frac{\partial}{\partial u_j} \left( \frac{\beta_{kj}}{\beta_{ij}} \right) \tag{26}$$

$$\mathcal{L}_{ij}(\beta_{kl}) = \beta_{kl} - \frac{\beta_{kj} \beta_{il}}{\beta_{ij}} \tag{27}$$



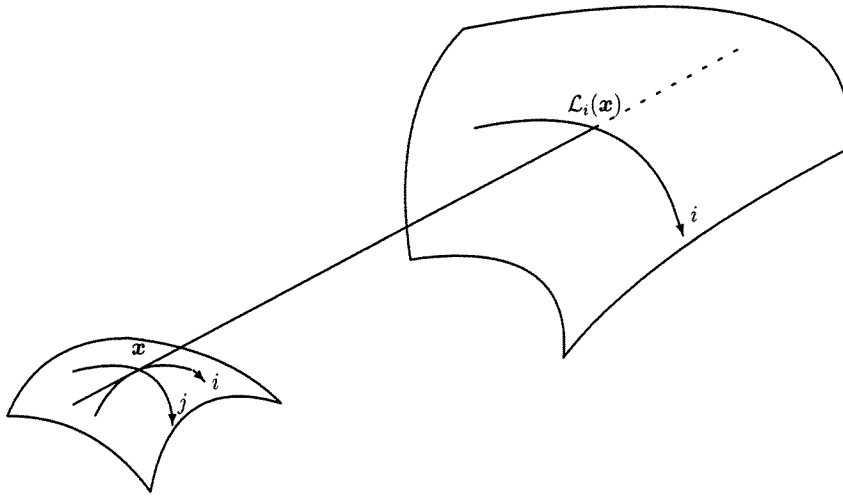


Figure 2. The Lévy transformation.

where all the indices,  $i, j, k$  and  $l$ , are different. It can be also shown that the Laplace transformations satisfy the following relations [14]

$$\mathcal{L}_{ij} \circ \mathcal{L}_{ji} = \mathcal{L}_{ji} \circ \mathcal{L}_{ij} = \text{id} \tag{28}$$

$$\mathcal{L}_{ij} \circ \mathcal{L}_{jk} = \mathcal{L}_{ik} \quad \mathcal{L}_{ij} \circ \mathcal{L}_{ki} = \mathcal{L}_{kj}. \tag{29}$$

We finally recall that the Laplace transformation  $\mathcal{L}_{ij}$  of a two-dimensional conjugate net provides the geometric meaning [7] of the two-dimensional Toda system. In fact, interpreting in equations (22) and (28) the operator  $\mathcal{L}_{ij}$  as translation in the discrete variable  $n$ , we obtain

$$\frac{\partial^2 \log \beta_{ij}(n)}{\partial u_i \partial u_j} = \frac{\beta_{ij}(n)}{\beta_{ij}(n-1)} - \frac{\beta_{ij}(n+1)}{\beta_{ij}(n)}.$$

(ii) The Lévy transformation  $\mathcal{L}_i(x)$  of a conjugate net  $x$  is a net conjugate to the  $i$ th tangent congruence [13]; i.e., the lines  $\langle x, \mathcal{L}_i(x) \rangle$  are *tangent* to the  $i$ th coordinate lines at  $x$  (see figure 2).

The position points of the new net are given by [13, 11]

$$\mathcal{L}_i(x) = x - \frac{\Omega[\zeta, H]}{\zeta_i} X_i \tag{30}$$

where the *transformation data*  $\zeta_k, k = 1, \dots, N$ , are solutions of the linear system (12), and  $\Omega[\zeta, H]$  is a solution of the equations

$$\frac{\partial \Omega}{\partial u_k} = \zeta_k H_k \quad k = 1, \dots, N.$$

The corresponding transformations for the tangent vectors  $X_i$  are [24]

$$\begin{aligned} \mathcal{L}_i(X_i) &= -\frac{\partial X_i}{\partial u_i} + \frac{1}{\zeta_i} \frac{\partial \zeta_i}{\partial u_i} X_i \\ \mathcal{L}_i(X_k) &= X_k - \frac{\zeta_k}{\zeta_i} X_i \quad k \neq i \quad k = 1, \dots, N. \end{aligned} \tag{31}$$

(iii) The adjoint Lévy transformation  $\mathcal{L}_i^*(x)$  of a conjugate net  $x$  is the  $i$ th focal net of a congruence conjugate to  $x$  [13]; i.e., the lines  $\langle x, \mathcal{L}_i^*(x) \rangle$  are *tangent* to the  $i$ th coordinate

lines of the new net. The position points of the new net are given by [13, 11]

$$\mathcal{L}_i^*(x) = x - \frac{\Omega[\mathbf{X}, \zeta^*]}{\zeta_i^*} H_i$$

where  $\zeta_k^*, k = 1, \dots, N$ , is a solution to the adjoint linear system (13) and  $\Omega[\mathbf{X}, \zeta^*]$  is a solution of the equations

$$\frac{\partial \Omega}{\partial u_k} = \mathbf{X}_k \zeta_k^* \quad k = 1, \dots, N.$$

The corresponding transformations for the tangent vectors  $\mathbf{X}_i$  are [24]

$$\begin{aligned} \mathcal{L}_i^*(\mathbf{X}_i) &= -\frac{\Omega[\mathbf{X}, \zeta^*]}{\zeta_i^*} \\ \mathcal{L}_i^*(\mathbf{X}_k) &= \mathbf{X}_k - \beta_{ki} \frac{\Omega[\mathbf{X}, \zeta^*]}{\zeta_i^*} \quad k \neq i \quad k = 1, \dots, N. \end{aligned} \tag{32}$$

(iv) The fundamental transformation  $\mathcal{F}(x)$  of a conjugate net  $x$  shares with  $x$  the same conjugate congruence; i.e., the lines  $\langle x, \mathcal{F}(x) \rangle$  intersect both nets along the coordinate lines. It can be viewed as the composition of Lévy and adjoint Lévy transformations  $\mathcal{F}_i = \mathcal{L}_i \circ \mathcal{L}_i^*$ . Notice that  $\mathcal{F}_i = \mathcal{F}_j$  for all  $i, j$ . The fundamental transformation is given by [13]

$$\mathcal{F}_i(x) = x - \Omega[\mathbf{X}, \zeta^*] \frac{\Omega[\zeta, H]}{\Omega[\zeta, \zeta^*]}$$

here  $\Omega[\zeta, \zeta^*]$  is a solution of

$$\frac{\partial \Omega}{\partial u_k} = \zeta_k \zeta_k^* \quad k = 1, \dots, N.$$

The corresponding transformations for the tangent vectors  $\mathbf{X}_i$  are [24]

$$\mathcal{F}_i(\mathbf{X}_j) = \mathbf{X}_j - \frac{\Omega[\mathbf{X}, \zeta^*]}{\Omega[\zeta, \zeta^*]} \zeta_j \quad j = 1, \dots, N. \tag{33}$$

We first remark that Lévy, adjoint Lévy and Laplace transformations are limiting cases of the fundamental transformation, in which one of the two nets (or both nets) conjugate to the congruence of the transformation are focal nets of the congruence [11].

We also remark that, from proposition 1, these transformations map solutions of the multicomponent KP hierarchy into new solutions and, in this context, they were recently investigated in [31], under the collective name of ‘Darboux transformations’, commonly used in the soliton community [29, 24].

It is a common belief that Darboux-type transformations of integrable partial differential equations generate their natural integrable discrete versions [27]. Furthermore, if the original partial differential equation has a geometric meaning, the Darboux-type transformations provide the natural discretization of the corresponding geometric notions [4, 25, 11]. For example, if we consider a conjugate net  $x$  and two fundamental transformations  $\mathcal{F}_1(x)$  and  $\mathcal{F}_2(x)$  of it, the points

$$\{x, \mathcal{F}_1(x), \mathcal{F}_2(x), \mathcal{F}_1(\mathcal{F}_2(x))\}$$

are coplanar [13]. It turns out that a lattice

$$\begin{aligned} x &: \mathbb{Z}^N \rightarrow \mathbb{R}^M \\ n &\mapsto x(n) \end{aligned}$$

$N \leq M$ , whose elementary quadrilaterals are planar (i.e., a quadrilateral lattice) is the correct discrete analogue of a conjugate net [34, 10]. The planarity condition can be expressed by the following linear equation (cf (18))

$$\Delta_j \mathbf{X}_i = (T_j Q_{ij}) \mathbf{X}_j \quad i, j = 1, \dots, N \quad i \neq j$$

being its compatibility conditions

$$\Delta_k Q_{ij} = (T_k Q_{ik}) Q_{kj} \quad i, j \text{ and } k \text{ different} \quad (34)$$

the discrete analogue of the Darboux equations (14). The points  $x$  of the lattice can be found by means of discrete integration of

$$\Delta_i x = (T_i H_i) X_i \quad i = 1, \dots, N$$

where  $H_i$  are solutions of equations

$$\Delta_i H_j = Q_{ij} T_i H_i \quad i, j = 1, \dots, N \quad i \neq j.$$

In the above formulae,  $T_i$  is the translation operator in the discrete variable  $n_i$ :

$$T_i f(n_1, \dots, n_i, \dots, n_N) = f(n_1, \dots, n_i + 1, \dots, n_N)$$

and  $\Delta_i = T_i - 1$  is the corresponding partial difference operator.

#### 4. Vertex operators as classical transformations of conjugate nets

In this section we show how the basic vertex operators associated with the multicomponent KP hierarchy have a natural geometrical interpretation as the classical transformations of conjugate nets.

##### 4.1. Identification between partial charge transformations in the zero charge sector and Laplace transformations

We first notice that the algebraic relations (28), (29) between the Laplace transformations are the same as those satisfied by the  $A_{N-1}$  root lattice operators shifts (the Schlesinger transformations)  $\mathcal{S}_{ij}$  (3), (4). In fact, both transformations can be identified as stated in the following.

**Proposition 2.** *The root lattice shift  $\mathcal{S}_{ij}$  in the direction  $\alpha_{ij}$  is the composition of the Laplace transformation  $\mathcal{L}_{ij}$  with a trivial scaling symmetry of the Darboux equations.*

**Proof.** Let us examine the Laplace transformation at the light of  $\tau$ -functions, by recalling the definition of the rotation coefficients (8). Starting from (21) we obtain  $\mathcal{L}_{ij} \beta_{ij} = -\varepsilon_{ij} \beta_{ij} \partial^2 \log \tau_{ij} / \partial u_i \partial u_j$ , where we have used (15) in the form  $\tau^2 \partial^2 \log \tau / \partial u_i \partial u_j = \tau_{ij} \tau_{ji}$ . If in the previous identity we apply  $\mathcal{S}_{ij}$  we get  $\tau_{ij}^2 \partial^2 \log \tau_{ij} / \partial u_i \partial u_j = (\mathcal{S}_{ij} \tau_{ij}) \tau$ , so that

$$\mathcal{L}_{ij} \beta_{ij} = -\mathcal{S}_{ij} \beta_{ij}.$$

For the next three equations (22)–(24) the following identifications trivially hold

$$\mathcal{L}_{ij} \beta_{ji} = -\mathcal{S}_{ij} \beta_{ji}$$

$$\mathcal{L}_{ij} \beta_{ki} = \varepsilon_{ki} \varepsilon_{kj} \varepsilon_{ij} \mathcal{S}_{ij} \beta_{ki}$$

$$\mathcal{L}_{ij} \beta_{jk} = \varepsilon_{ki} \varepsilon_{kj} \varepsilon_{ji} \mathcal{S}_{ij} \beta_{jk}.$$

The next two equations (25) and (26) derive from equation (16) once the shifts  $\mathcal{S}$  are applied properly; namely, we have the identities

$$\tau_{ik} \frac{\partial \tau_{ij}}{\partial u_i} - \tau_{ij} \frac{\partial \tau_{ik}}{\partial u_i} = \varepsilon_{kj} \varepsilon_{ki} \varepsilon_{ij} \tau \mathcal{S}_{ik} \tau_{ij}$$

$$\tau_{kj} \frac{\partial \tau_{ij}}{\partial u_j} - \tau_{ij} \frac{\partial \tau_{kj}}{\partial u_j} = \varepsilon_{kj} \varepsilon_{ki} \varepsilon_{ij} \tau \mathcal{S}_{kj} \tau_{ij}$$

from where it follows

$$\begin{aligned} \mathcal{L}_{ij}\beta_{ik} &= \varepsilon_{ki}\varepsilon_{kj}\varepsilon_{ij}\mathcal{S}_{ij}\beta_{ik} \\ \mathcal{L}_{ij}\beta_{kj} &= \varepsilon_{ki}\varepsilon_{kj}\varepsilon_{ji}\mathcal{S}_{ij}\beta_{kj}. \end{aligned}$$

Finally, (17) leads to

$$\mathcal{L}_{ij}\beta_{kl} = \varepsilon_{ki}\varepsilon_{kj}\varepsilon_{li}\varepsilon_{lj}\mathcal{S}_{ij}\beta_{kl}.$$

These results can be summarized as

$$\frac{a_k}{a_l}\mathcal{S}_{ij}\beta_{kl} = \mathcal{L}_{ij}\beta_{kl} \quad k, l = 1, \dots, N \quad k \neq l$$

where:

$$a_k = \varepsilon_{ki}\varepsilon_{kj}.$$

At this point we must remark that the Darboux equations have the following scaling symmetry  $\beta_{ij} \rightarrow a_j/a_i \beta_{ij}$  that comes from the freedom in the choice of the Lamé coefficients  $H_i \rightarrow a_i H_i$ , where  $a_i = a_i(u_i)$ ,  $i = 1, \dots, N$ , are functions of  $u_i$  only.  $\square$

Thus, a  $\tau$ -function of the multicomponent KP hierarchy describes not only the integrable deformations of a single conjugate net but also all its Laplace transforms.

#### 4.2. Vertex operators $\mathbb{G}_i$ and $\mathbb{G}_i^*$ as Lévy transformations

Now we consider the action of the basic vertex operators at the level of Baker functions. In the next proposition we identify the action of the vertex operator  $\mathbb{G}_i(p)$  with the the classical Lévy transformation.

**Proposition 3.** *Given tangent vectors  $X_j$ ,  $i = j, \dots, M$ , associated with the Baker function  $\psi(z, t)$  as prescribed in (19), then*

$$\mathcal{L}_i(X_j) = p^{\delta_{ij}}\mathbb{V}_i^-(p)(X_j)$$

where  $\mathcal{L}_i$  stands for the Lévy transformation with data

$$\zeta_j(t) := \frac{d^n \psi_{ji}}{dz^n}(p, t) \quad j = 1, \dots, M$$

where  $n \geq 0$  is the order of the first non-zero  $z$ -derivative of the  $i$ th column of  $\psi(z, t)$  at  $z = p$ .

**Proof.** First, we observe that the following asymptotic expansion holds:

$$\begin{aligned} \mathbb{V}_i^-(p)\psi(z, t) &= [\mathbb{V}_i^-(p)\chi(z, t)] \left(1 - \frac{z}{p}P_i\right) \psi_0(z, t) \\ &= \left[-\frac{z}{p}P_i + 1 - \frac{1}{p}\mathbb{V}_i^-(\beta P_i) + \mathcal{O}\left(\frac{1}{z}\right)\right] \psi_0(z, t) \end{aligned} \tag{35}$$

which can be compared with

$$\frac{\partial \psi}{\partial u_i} = \left[zP_i + \beta P_i + \mathcal{O}\left(\frac{1}{z}\right)\right] \psi_0(z, t).$$

Thus, as both  $\mathbb{V}_i^-(p)\psi$  and  $\partial \psi / \partial u_i$  belong to the Grassmannian element  $W$  and taking into account (10) and (11) we deduce

$$\mathbb{V}_i^-(p)\psi = -\frac{1}{p} \frac{\partial \psi}{\partial u_i} + \left(1 - \frac{1}{p}(\mathbb{V}_i^-(p) - 1)\beta P_i\right) \psi. \tag{36}$$

Moreover, by using the matrix form of equations (14)

$$P_j \frac{\partial \psi}{\partial u_i} = P_j \beta P_i \psi \quad i \neq j$$

one gets

$$\frac{\partial \psi}{\partial u_i} = P_i \frac{\partial \psi}{\partial u_i} + \sum_{j \neq i} P_j \beta P_i \psi = P_i \frac{\partial \psi}{\partial u_i} + (\beta P_i - P_i \beta P_i) \psi.$$

Hence (36) becomes

$$\mathbb{V}_i^-(p) \psi = \psi - \frac{1}{p} P_i \frac{\partial \psi}{\partial u_i} + \frac{1}{p} P_i \beta P_i \psi - \frac{1}{p} (\mathbb{V}_i^-(p) \beta P_i) \psi$$

or equivalently

$$P_j \mathbb{V}_i^-(p) \psi = P_j \psi - \frac{1}{p} (\mathbb{V}_i^-(p) \beta_{ji} E_{ji}) \psi \quad i \neq j \tag{37}$$

and

$$P_i \mathbb{V}_i^-(p) \psi = -\frac{1}{p} P_i \frac{\partial \psi}{\partial u_i} + f(t) P_i \psi \tag{38}$$

where

$$f(t) := 1 + \frac{1}{p} (1 - \mathbb{V}_i^-(p)) \beta_{ii}.$$

From (35) we notice that  $\mathbb{V}_i^-(p) \psi(p, t) P_i = 0$ , so that by setting  $z = p$  in (37) and (38) we conclude

$$\begin{aligned} \frac{1}{p} (\mathbb{V}_i^-(p) \beta_{ji})(t) &= \frac{\zeta_j(t)}{\zeta_i(t)} \\ f(t) &= \frac{1}{p} \frac{\partial \log \zeta_i}{\partial u_i}(p, t) \end{aligned}$$

where

$$\zeta_j(t) := \frac{d^n \psi_{ji}}{dz^n}(p, t). \tag{39}$$

Therefore, we may rewrite (37) and (38) as

$$\begin{aligned} (\mathbb{V}_i^-(p) \psi_j)(z, t) &:= \psi_j(z, t) - \frac{\zeta_j(t)}{\zeta_i(t)} \psi_i(z, t) \quad j \neq i \\ (\mathbb{V}_i^-(p) \psi_i)(z, t) &:= -\frac{1}{p} \frac{\partial \psi_i}{\partial u_i}(z, t) + \frac{1}{p} \frac{\partial \log \zeta_i}{\partial u_i}(t) \psi_i(z, t). \end{aligned} \tag{40}$$

The rows of  $\psi$  and  $\mathbb{V}_i^-(p) \psi$  provide tangent vectors for conjugate nets, so that by comparing (40) with (31) we obtain

$$\mathcal{L}_i(\psi_j) = p^{\delta_{ij}} \mathbb{V}_i^-(\psi_j).$$

Hence, from (19) we get the desired result. □

Now we identify the vertex operator  $\mathbb{G}_i^*(p)$  with the adjoint Lévy transformation

**Proposition 4.** *Given tangent vectors  $X_j, j = 1, \dots, M$ , associated with the Baker function as prescribed in (19), then*

$$\mathcal{L}_i^*(X_j) = \frac{1}{p^{\delta_{ij}}} \mathbb{V}_i^+(X_j)$$

where  $\mathcal{L}_i^*$  stands for the adjoint Lévy transformation with data

$$\zeta_j^*(\mathbf{t}) = \frac{d^m \psi_{ij}^*}{dz^m}(p, \mathbf{t}) \quad j = 1, \dots, M$$

with  $m$  being the order of the first non-vanishing  $z$ -derivative of the  $i$ th row of  $\psi^*(z, \mathbf{t})$  at  $z = p$ , and potential

$$\Omega = -\frac{1}{p}[\mathbb{V}_i^+(p)\psi_i]\zeta_i^*.$$

**Proof.** On the one hand, setting  $\mathbf{t}' = \mathbf{t} - [1/p]\mathbf{e}_i$  in the bilinear identity (9) one gets

$$(1 - P_i)\mathbb{V}_i^-(p)\beta - \beta(1 - P_i) + pP_i - p\chi(p, \mathbf{t})P_i\mathbb{V}_i^-(p)\chi^*(p, \mathbf{t}) = 0$$

which implies

$$\begin{aligned} \beta_{ji}(\mathbf{t}) &= p[\mathbb{V}_i^+(p)\chi_{ji}(p, \mathbf{t})]\chi_{ii}^*(p, \mathbf{t}) \\ \beta_{ij}(\mathbf{t}) &= -p[\mathbb{V}_i^-(p)\chi_{ij}^*(p, \mathbf{t})]\chi_{ii}(p, \mathbf{t}) \\ \chi_{ii}^*(p, \mathbf{t})\mathbb{V}_i^+(p)\chi_{ii}(p, \mathbf{t}) &= 1 \end{aligned}$$

for  $j = 1, \dots, N$  and  $i \neq j$ ; hence,

$$\begin{aligned} \frac{[\mathbb{V}_i^+(p)\psi_{ji}](p, \mathbf{t})}{[\mathbb{V}_i^+(p)\psi_{ii}](p, \mathbf{t})} &= \lim_{z \rightarrow p} \frac{[\mathbb{V}_i^+(p)\psi_{ji}](z, \mathbf{t})}{[\mathbb{V}_i^+(p)\psi_{ii}](z, \mathbf{t})} = \frac{[\mathbb{V}_i^+(p)\chi_{ji}](p, \mathbf{t})}{[\mathbb{V}_i^+(p)\chi_{ii}](p, \mathbf{t})} = \frac{\beta_{ji}(\mathbf{t})\chi_{ii}^*(p, \mathbf{t})}{p\chi_{ii}^*(p, \mathbf{t})} \\ &= \frac{1}{p}\beta_{ji}(\mathbf{t}) \end{aligned}$$

$$\begin{aligned} \frac{[\mathbb{V}_i^-(p)\psi_{ij}^*](p, \mathbf{t})}{[\mathbb{V}_i^-(p)\psi_{ii}^*](p, \mathbf{t})} &= \lim_{z \rightarrow p} \frac{[\mathbb{V}_i^-(p)\psi_{ij}^*](z, \mathbf{t})}{[\mathbb{V}_i^-(p)\psi_{ii}^*](z, \mathbf{t})} = \frac{[\mathbb{V}_i^-(p)\chi_{ij}^*](p, \mathbf{t})}{[\mathbb{V}_i^-(p)\chi_{ii}^*](p, \mathbf{t})} = -\frac{\beta_{ij}(\mathbf{t})\chi_{ii}(p, \mathbf{t})}{p\chi_{ii}(p, \mathbf{t})} \\ &= -\frac{1}{p}\beta_{ij}(\mathbf{t}) \end{aligned}$$

$$\frac{\partial \log \mathbb{V}_i^+(p)\psi_{ii}}{\partial u_i}(p, \mathbf{t}) = \lim_{z \rightarrow p} \frac{\partial \log \mathbb{V}_i^+(p)\psi_{ii}(z, \mathbf{t})}{\partial u_i} = -\frac{\partial \log \psi_{ii}^*}{\partial u_i}(p, \mathbf{t})$$

for  $j = 1, \dots, N$  and  $i \neq j$ . By using l'Hôpital rule with  $\zeta_j(\mathbf{t}) := \frac{d^n \psi_{ji}}{dz^n}(p, \mathbf{t})$  and  $\zeta_j^*(\mathbf{t}) := \frac{d^m \psi_{ij}^*}{dz^m}(p, \mathbf{t})$ , where  $n$  and  $m$  are the orders of first non-vanishing derivatives of the  $z$ -derivatives of the  $i$ th column of  $\psi$  and  $i$ th row of  $\psi^*$ , respectively, we get the identities:

$$\begin{aligned} \frac{\mathbb{V}_i^+(p)\zeta_j}{\mathbb{V}_i^+(p)\zeta_i} &= \frac{1}{p}\beta_{ji} \\ \frac{\mathbb{V}_i^-(p)\zeta_j^*}{\mathbb{V}_i^-(p)\zeta_i^*} &= -\frac{1}{p}\beta_{ij} \\ \frac{\partial \log \mathbb{V}_i^+(p)\zeta_i}{\partial u_i} &= -\frac{\partial \log \zeta_i^*}{\partial u_i} \end{aligned} \tag{41}$$

with  $j = 1, \dots, N$  and  $i \neq j$ .

On the other hand, from (40) it follows that

$$\begin{aligned} \psi_j &= \mathbb{V}_i^+(p)\psi_j - \frac{\mathbb{V}_i^+(p)\zeta_j}{\mathbb{V}_i^+(p)\zeta_i}\mathbb{V}_i^+(p)\psi_i \quad j \neq i \\ \psi_i &= -\frac{1}{p}\frac{\partial \mathbb{V}_i^+(p)\psi_i}{\partial u_i} + \frac{1}{p}\frac{\partial \log \mathbb{V}_i^+(p)\zeta_i}{\partial u_i}\mathbb{V}_i^+(p)\psi_i. \end{aligned}$$

Therefore, these relations become

$$\begin{aligned}\psi_j &= \mathbb{V}_i^+(p)\psi_j - \frac{1}{p}\beta_{ji}\mathbb{V}_i^+(p)\psi_i & j \neq i \\ \psi_i &= -\frac{1}{p}\frac{1}{\zeta_i^*}\frac{\partial \zeta_i^* \mathbb{V}_i^+(p)\psi_i}{\partial u_i}.\end{aligned}\quad (42)$$

In order to identify them with the Lévy transformations it is required to introduce the potential

$$\Omega := -\frac{1}{p}[\mathbb{V}_i^+(p)\psi_i]\zeta_i^*.$$

The second equation in (42) can be written as  $\frac{\partial \Omega}{\partial u_i} = \psi_i \zeta_i^*$ ; now, we proceed to identify the other partial derivatives:

$$\begin{aligned}-p\frac{\partial \Omega}{\partial u_j} &= [\mathbb{V}_i^+(p)\beta_{ij}][\mathbb{V}_i^+(p)\psi_j]\zeta_i^* + \beta_{ji}[\mathbb{V}_i^+(p)\psi_i]\zeta_j^* \\ &= -p[\mathbb{V}_i^+(p)\psi_j]\zeta_j^* - p(\psi_j - \mathbb{V}_i^+(p)\psi_j)\zeta_j^* \\ &= -p\psi_j\zeta_j^* & j \neq i\end{aligned}$$

where we have used equations (41) and (42). Thus, we deduce that  $\Omega$  is characterized up to a constant vector by

$$\frac{\partial \Omega}{\partial u_j} = \psi_j \zeta_j^* \quad j = 1, \dots, N.$$

With the aid of  $\Omega$  we express (42) as

$$\begin{aligned}\mathbb{V}_i^+(p)\psi_j &= \psi_j - \beta_{ji}\frac{\Omega}{\zeta_i^*} & j \neq i \\ \mathbb{V}_i^+(p)\psi_i &= -p\frac{\Omega}{\zeta_i^*}.\end{aligned}$$

Therefore, by comparing with (32) we have

$$\mathcal{L}_i^*(\psi_j) = \frac{1}{p^{\delta_{ij}}}\mathbb{V}_i^+(p)(\psi_j)$$

where  $\mathcal{L}_i^*$  stands for the adjoint Lévy transformation with data  $\zeta_j^*$ . □

### Remarks.

- (1) For the one component KP hierarchy our results for the Lévy transformations reduces to those in [1] for the Darboux transformations.
- (2) Notice that we might consider integer powers of the vertex operators  $\mathbb{V}_i^-(p)$ , say  $\mathbb{V}_i^-(p)^{n_i}$ , that models the shift  $f(\mathbf{t}) \rightarrow f(\mathbf{t} - n_i[1/p]\mathbf{e}_i)$ . From our proposition 3 is clear that it can be thought of as a sequence of Lévy transformations, that we will use in section 5. However, we stress that, even when the initial Baker function  $\psi$  does not vanish at  $p$ , its transformed function does; hence, we should take its  $z$ -derivative at  $p$  to get the new transformation data. Thus, the sequence is defined in terms of the truncated jet of the initial Baker function:

$$\left\{ \psi_{ji}(p, \mathbf{t}), \frac{d\psi_{ji}}{dz}(p, \mathbf{t}), \dots, \frac{d^{n_i-1}\psi_{ji}}{dz^{n_i-1}}(p, \mathbf{t}) \right\}.$$

(3) In matrix terms the above propositions can be summarized as

$$\begin{aligned} \mathcal{L}_i(\psi) &= p^{P_i} [\mathbb{G}_i(p)\psi] \left(-\frac{z}{p}\right)^{P_i} \\ \mathcal{L}_i^*(\psi) &= \left(\frac{1}{p}\right)^{P_i} [\mathbb{G}_i^*(p)\psi] \left(-\frac{p}{z}\right)^{P_i}. \end{aligned}$$

(4) We notice that the correspondences provided by the last two propositions, derived from the bilinear equation for Baker functions, are direct consequences, when  $i \neq j$ , of Fay identities for the  $\tau$ -function. Namely, for the Lévy transformation the relevant Fay identity is

$$\begin{aligned} \varepsilon_{jk} z^{\delta_{jk}-1} p^{\delta_{ki}} \tau_{jk}(\mathbf{t} - [1/z]e_k) \tau(\mathbf{t} - [1/p]e_i) \\ + \varepsilon_{ki} \varepsilon_{ji} z^{\delta_{ik}-1} p^{\delta_{ki}+\delta_{ij}-1} \tau_{ik}(\mathbf{t} - [1/z]e_k) \tau_{ji}(\mathbf{t} - [1/p]e_i) \\ - \varepsilon_{jk} z^{\delta_{jk}-1} (p-z)^{\delta_{ki}} \tau_{jk}(\mathbf{t} - [1/z]e_k - [1/p]e_i) \tau(\mathbf{t}) = 0 \quad i \neq j. \end{aligned}$$

For the adjoint Lévy transformation the corresponding Fay identity is

$$\begin{aligned} \varepsilon_{jk} z^{\delta_{jk}-1} (p-z)^{\delta_{ki}} \tau_{jk}(\mathbf{t} - [1/z]e_k) \tau(\mathbf{t} + [1/p]e_i) \\ - \varepsilon_{ki} \varepsilon_{ji} z^{\delta_{ik}-1} p^{\delta_{ki}-1} \tau_{ik}(\mathbf{t} - [1/z]e_k + [1/p]e_i) \tau_{ji}(\mathbf{t}) \\ - \varepsilon_{jk} z^{\delta_{jk}-1} p^{\delta_{ik}} \tau_{jk}(\mathbf{t} - [1/z]e_k + [1/p]e_i) \tau(\mathbf{t}) = 0 \quad i \neq j. \end{aligned}$$

### 4.3. The soliton vertex operator as fundamental transformation

In the context of the  $\tau$ -function theory, the soliton solutions are generated by composite vertex operators which are infinitesimally generated by  $b_i(p)c_j(q)$ . It what follows we will concentrate on the diagonal case  $i = j$ :

$$\Gamma_i(p, q)\tau(\ell, \mathbf{t}) := \left(\frac{p}{q}\right)^{\ell_i} \exp(\xi(p, \mathbf{t}_i) - \xi(q, \mathbf{t}_i))\tau(\ell, \mathbf{t} - [1/p]e_i + [1/q]e_i).$$

Since  $\Gamma_i^2 = 0$  the exponential action reduces to

$$\exp(aX_i(p, q)) = 1 + \Gamma_i(p, q).$$

We shall show here that it correspond to a fundamental transformation. To this end we need the following proposition.

**Proposition 5.** *The  $\tau$ -function satisfies the following identities for any  $i, j, k \in 1, \dots, N$ :*

$$\begin{aligned} \varepsilon_{jk} z^{\delta_{jk}-1} \frac{\tau_{jk}(\mathbf{t} - [1/z]e_k)}{\tau(\mathbf{t})} \\ - \varepsilon_{ik} \varepsilon_{ji} z^{\delta_{ik}-1} p^{\delta_{ji}-1} \frac{1 - \frac{p}{q}}{\left(1 - \frac{z}{q}\right)^{\delta_{ki}}} \frac{\tau_{ik}(\mathbf{t} - [1/z]e_k + [1/q]e_i)}{\tau(\mathbf{t} - [1/p]e_i + [1/q]e_i)} \frac{\tau_{ji}(\mathbf{t} - [1/p]e_i)}{\tau(\mathbf{t})} \\ = \varepsilon_{jk} z^{\delta_{jk}-1} \left(\frac{p}{q}\right)^{\delta_{ij}} \left(\frac{1 - \frac{z}{p}}{1 - \frac{z}{q}}\right)^{\delta_{ki}} \frac{\tau_{jk}(\mathbf{t} - [1/z]e_k - [1/p]e_i + [1/q]e_i)}{\tau(\mathbf{t} - [1/p]e_i + [1/q]e_i)}. \end{aligned} \quad (43)$$

**Proof.** On the one hand, as one can readily check from (5), the right-hand side of (43) is, up to exponential factors, just the  $\tau$ -function representation of the components of the following vector

$$\left(\frac{p}{q}\right)^{\delta_{ij}} \mathbb{V}_i^-(p)\mathbb{V}_i^+(q)\psi_j.$$



On the other hand, we know that this is the composition of an adjoint Lévy transformation, with transforming function  $\zeta_j^*(\mathbf{t}) = \psi_{ij}^*(q, \mathbf{t})$  and potential  $\Omega(z) = -\frac{1}{q}[\mathbb{V}_i^+(q)\psi_i(z)]\zeta_i^*$ , and a Lévy transformation with data  $\zeta_j(\mathbf{t}) = (1 - p/q)^{\delta_{ij}}\psi_{ji}(p, \mathbf{t})$  (the Baker function after the first adjoint Lévy transformation is obtained from  $\mathbb{V}_i^+(q)\psi$  by suitable normalization). Such a composition is a fundamental transformation:

$$\mathcal{F}_i(\psi_j) = \psi_j - \frac{\Omega}{\zeta_j} \quad j = 1, \dots, N$$

where

$$\begin{aligned} \Omega &= -\frac{1}{q}[\mathbb{V}_i^+(q)\psi_i]\zeta_i^* \\ \Omega &= -\frac{1}{q}[\mathbb{V}_i^+(q)\zeta_i]\zeta_i^*. \end{aligned}$$

That is to say, the matrix elements of the Baker function satisfy

$$\psi_{jk}(z) - \frac{\mathbb{V}_i^+(q)\psi_{ik}(z)}{\mathbb{V}_i^+(q)\psi_{ii}(p)}\psi_{ji}(p) = \left(\frac{p}{q}\right)^{\delta_{ij}}\mathbb{V}_i^-(p)\mathbb{V}_i^+(q)\psi_{jk}(z) \quad i, j, k = 1, \dots, N.$$

After substituting the expression (5) of the Baker function in terms of the  $\tau$ -function we obtain the desired identities.  $\square$

From (43) one arrives to the following result.

**Proposition 6.** *Given tangent vectors  $\mathbf{X}_j$ ,  $i = j, \dots, M$ , associated with the Baker function  $\psi(z, \mathbf{t})$  as prescribed in (19), the induced action of the operator  $\exp(aX_i(p, q))$  is given by the fundamental transformation*

$$\mathcal{F}_i(\mathbf{X}_j) = \mathbf{X}_j - \frac{\Omega}{\zeta_j} \quad j = 1, \dots, N$$

with transforming data  $\zeta_j(\mathbf{t}) = (1 - p/q)\psi_{ji}(p, \mathbf{t})$  and  $\zeta_j^*(\mathbf{t}) = \psi_{ij}^*(q, \mathbf{t})$ ,  $j = 1, \dots, N$ , and potentials:

$$\begin{aligned} \Omega &= -\frac{1}{q}[\mathbb{V}_i^+(q)\mathbf{X}_i]\zeta_i^* \\ \Omega &= -\frac{1}{a} \frac{q^{\ell_i-1}}{p^{\ell_i}} - \frac{1}{q}[\mathbb{V}_i^+(q)\zeta_i]\zeta_i^*. \end{aligned}$$

**Proof.** In the expression (5) substitute the old  $\tau$ -function by the new one  $(1 + a\Gamma_i(p, q))\tau$ , paying particular attention, in the numerator, to the action of  $\mathbb{V}_k^-(z)$  on this new function. Then, using (43), the definition (5) and comparing with (33) we obtain the desired result.  $\square$

### Remarks.

- (1) In the last two propositions we are assuming that  $p$  and  $q$  are generic points for the corresponding Baker functions; i.e.  $p$  and  $q$  are not zeroes of  $\psi$  and  $\psi^*$ , respectively.
- (2) An alternative derivation of (43) follows from the bilinear equation (2) by choosing  $\mathbf{t}'$  and  $\ell'$  appropriately and evaluating the corresponding residues of the integrand. In fact they constitute a typical set of Fay identities:

$$\begin{aligned} &\varepsilon_{jk}z^{\delta_{jk}-1}p^{\delta_{ki}-1}q^{\delta_{ij}-1}(z-q)^{\delta_{ki}}\tau_{jk}(\mathbf{t} - [1/z]\mathbf{e}_k)\tau(\mathbf{t} - [1/p]\mathbf{e}_i + [1/q]\mathbf{e}_i) \\ &\quad - \varepsilon_{ki}\varepsilon_{ji}z^{\delta_{ik}-1}p^{\delta_{ki}-1}q^{\delta_{ki}+\delta_{ij}-2}(p-q)\tau_{ik}(\mathbf{t} - [1/z]\mathbf{e}_k \\ &\quad + [1/q]\mathbf{e}_i)\tau_{ji}(\mathbf{t} - [1/p]\mathbf{e}_i) - \varepsilon_{jk}z^{\delta_{jk}-1}p^{\delta_{ij}-1}q^{\delta_{ki}-1}(z-p)^{\delta_{ki}}\tau_{jk} \\ &\quad \times (\mathbf{t} - [1/z]\mathbf{e}_k - [1/p]\mathbf{e}_i + [1/q]\mathbf{e}_i)\tau(\mathbf{t}) = 0. \end{aligned}$$

- (3) Observe the presence of the parameter  $a$  in the expression of the potential  $\Omega$ . It plays the role of an integration constant corresponding to the formula  $\frac{\partial \Omega}{\partial u_j} = \zeta_j \zeta_j^*$ ,  $j = 1, \dots, N$ .
- (4) This result strongly suggests a similar statement for the more general soliton operator:  $\exp(1 + ab_i(p)c_j(q))$ , but here we should have a composition of Lévy and adjoint Lévy in different directions and the potentials would have now general integration constants.

### 5. Miwa transformation and quadrilateral lattices

Let us consider the bilinear identity (9) for the Baker function  $\psi(z, \mathbf{t})$  and its adjoint  $\psi^*(z, \mathbf{t})$ . For each complex number  $p$  we can introduce new functions depending on  $N$  additional discrete variables,  $\mathbf{n} \in \mathbb{Z}^N$ , by defining

$$\Psi(z, \mathbf{t}, \mathbf{n}) := \psi(z, \mathbf{t} - \mathbf{n}[1/p]) \quad \Psi^*(z, \mathbf{t}, \mathbf{n}) := \psi^*(z, \mathbf{t} - \mathbf{n}[1/p])$$

where we understand that

$$\mathbf{t} - \mathbf{n}[1/p] = (t_1 - n_1[1/p], t_2 - n_2[1/p], \dots, t_N - n_N[1/p]).$$

Obviously (9) becomes a continuous-discrete bilinear equation of the form

$$\int_{S^1} dz \Psi(z, \mathbf{t}, \mathbf{n}) \Psi^*(z, \mathbf{t}', \mathbf{n}') = 0$$

for any  $\mathbf{t}, \mathbf{t}' \in \mathbb{C}^{N \cdot \infty}$  and  $\mathbf{n}, \mathbf{n}' \in \mathbb{Z}^N$ .

From (6) it follows that

$$\begin{aligned} \Psi(z, \mathbf{t}, \mathbf{n}) &:= \Xi(z, \mathbf{t}, \mathbf{n}) \Psi_0(z, \mathbf{t}, \mathbf{n}) \\ \Psi^*(z, \mathbf{t}, \mathbf{n}) &:= \Psi_0(z, \mathbf{t}, \mathbf{n})^{-1} \Xi^*(z, \mathbf{t}, \mathbf{n}) \end{aligned}$$

where

$$\Psi_0(z, \mathbf{t}, \mathbf{n}) := \psi_0(z, \mathbf{t}) \operatorname{diag} \left( \left(1 - \frac{z}{p}\right)^{n_1}, \dots, \left(1 - \frac{z}{p}\right)^{n_N} \right)$$

and

$$\begin{aligned} \Xi(z, \mathbf{t}, \mathbf{n}) &:= \chi(z, \mathbf{t} - \mathbf{n}[1/p]) \\ \Xi^*(z, \mathbf{t}, \mathbf{n}) &:= \chi^*(z, \mathbf{t} - \mathbf{n}[1/p]) \end{aligned}$$

have the following asymptotic expansion

$$\begin{aligned} \Xi(z) &\sim 1 - pQz^{-1} + \mathcal{O}(z^{-2}) & z \rightarrow \infty \\ \Xi^*(z) &\sim 1 + pQz^{-1} + \mathcal{O}(z^{-2}) & z \rightarrow \infty \end{aligned}$$

with

$$Q_{ij}(\mathbf{t}, \mathbf{n}) := -\frac{1}{p} \beta_{ij}(\mathbf{t} - \mathbf{n}[1/p]).$$

If we fix our attention on the  $\mathbf{n}$  dependence the asymptotic module structure is now

$$W = \bigoplus_{n \geq 0} M_N(\mathbb{C}) \cdot v_n(\mathbf{n}) \quad v_n(z, \mathbf{n}) = \left( \sum_{k=1}^N \Delta_k \right)^n \Psi(z, \mathbf{n}).$$

The linear systems for  $\Psi$  follow from the decomposition of the discrete derivatives of  $\Psi$  in terms of  $v_n$ . A similar analysis holds for  $\Psi^*$  and we obtain the following.

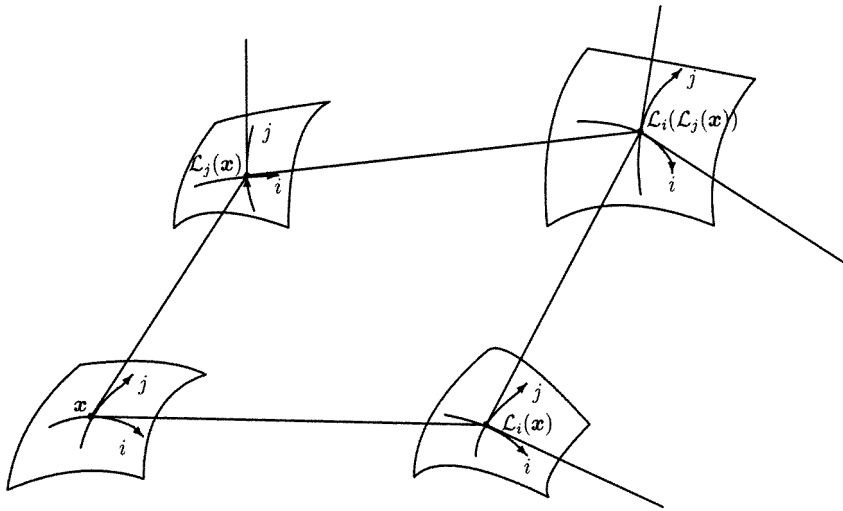


Figure 3. The quadrilateral lattice generated by Lévy transformations on a conjugate net.

**Proposition 7.** *The objects  $Q$ ,  $\Psi$  and  $\Psi^*$  do satisfy*

$$\Delta_k \Psi_i = (T_k Q_{ik}) \Psi_k \quad i \neq k \tag{44}$$

$$\Delta_k \Psi_j^* = Q_{kj} (T_k \Psi_k^*) \quad j \neq k. \tag{45}$$

**Remarks.**

- (1) Observe that (44) has been already proved in the first formula of (42), where one should apply  $\mathbb{V}_i^-(p)$  and perform the replacement  $\mathbb{V}_i^-(p) \rightarrow T_i$ .
- (2) The compatibility of (44) gives the discrete Darboux equation (34). It is clear that  $X_i(\mathbf{n})$  and  $H_i(\mathbf{n})$  can be obtained by the analogues of equations (19) and (20), respectively. Hence, we have a quadrilateral lattice in the discrete variable  $\mathbf{n}$ . From a geometrical point of view this has a clear interpretation. As we mentioned in section 3, the Darboux-type transformations of soliton equations provide their integrable discretization [27]. In the present Miwa-like scheme the translation  $T_i$  in the  $n_i$  variable corresponds to the vertex operator  $\mathbb{V}_i^-(p)$ . Since, from proposition 3,  $\mathbb{V}_i^-(p)$  corresponds to a Lévy transformation then  $\mathbf{x}(\mathbf{n})$  describes a quadrilateral lattice (see figure 3).
- (3) Obviously our approach gives, through the Miwa transformation, a  $\tau$ -function formulation of the quadrilateral lattices and a quantum field theoretical representation of them in terms of  $b$ - $c$  systems. For completeness, we give the  $\tau$ -function expression of the quadrilateral lattice equation (34):

$$\begin{aligned} (T_i \tau)(T_j \tau) - \tau(T_i T_j \tau) - (T_i \tau_{ij})(T_j \tau_{ji}) &= 0 & i \neq j \\ \tau(T_k \tau_{ij}) - (T_k \tau) \tau_{ij} - \varepsilon_{ij} \varepsilon_{ik} \varepsilon_{kj} (T_k \tau_{ik}) \tau_{kj} &= 0 & i, j \text{ and } k \text{ different.} \end{aligned}$$

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